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Stochastic Modeling for Abnormal Glucose Metabolism and Diabetes using Weibull Distribution

Dr N.Umamaheswari[1] and M. Dharshini[2] and R.Guna Sundhari[3]

[1] Assistant professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy, Tamil Nadu, India.

[2]M.Sc Mathematics , Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy , Tamil Nadu, India.

[3]M.Sc Mathematics , Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy , Tamil Nadu, India.

ABSTRACT

This study exposes the epidemiology of abnormal glucose metabolism namely: Diabetes and impaired fasting glucose (IFG) in 13 administrative regions. We present some methods for estimating Weibull distribution with parameters β (shape) and η (scale). This distribution is an important especially for reliability and maintainability analysis. Computational experiments on the presented methods are reported.

Keywords:

Weibull Probability, Hazard Plotting Technique, Maximum Likelihood Estimator (MLE), Methods of Moments (MOM), Least Square Method (LSM)

Introduction:

Abnormal glucose metabolism has reached an epidermic state in Saudi Arabia, where one-third of the population is affected and half of diabetic cases unaware of their disease. This observation warrants an urgent strategy for launching diabetes primary prevention and screening programs. In a country known for its wide geographical area and unique homogeneous social-cultural structure with rapid populations growth. Over the last 4 decades, this country has passes through a rapid economic development leading to urbanization reflected by life style changes.This life style change has led to soaring rates of chronic disease, mainly diabetes mellitus, which has put a real challenge to the health system in Saudi Arabia. The Weibull distribution is a probability distribution that is used to model failure times, analyze life data, and arsers reliability. It is a flexible tool that can be used in a variety of fields.

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Weibull Probability Plotting

The Weibull distribution density function (Mann et al. (1974) is given by:

$$f(x) = \frac{\beta}{\eta}\left(\frac{x-\beta}{\eta}\right)^{\beta-1} e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}}, \beta > 0, n > 0, x \geq \gamma \geq 0 \quad (1)$$

The cumulative Weibull distribution function is given by:

$$F(x) = 1 - e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}} \quad (2)$$

where; b is the shape parameter, η is the scale parameter, and γ is the location parameter. To come up with the relation between the CDF and the two parameters (β, η), we take the double logarithmic transformation of the CDF.

From (2) and letting g = 0, we have

$$1-F(x) = e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}}$$

$$\frac{1}{1-F(x)} = e^{-\left(\frac{x-\gamma}{\eta}\right)^{\beta}}$$

$$\ln\left[\frac{1}{1-F(x)}\right] = -\left(\frac{x}{\eta}\right)^{\beta}$$

$$\ln\ln\left[\frac{1}{1-F(x)}\right] = \beta \ln \eta - \beta \ln x$$

The last is an equation of a straight line. To plot F(x) versus x, we apply the following procedure:

1. Rank failure times in ascending order.
2. Estimate F(x_i) of the i^{th} failure'
3. Plot F (x_i) vs. x in the Weibull probability paper.

   To estimate F (x_i) in (2 and 3) above, we may use one of the following methods presented in Table 1 where n is number of data points.

Method	F (x_i)
Mean Rank	$\frac{i}{n+1}$
Median Rank	$\frac{i-0.3}{n+0.4}$
Symmetrical Rank	$\frac{i-0.5}{n}$

Table 1.Methods for estimating F (x_i) .

Hazard Plotting Technique

The hazard plotting technique is an estimation procedure for the Weibull parameters. This is done by plotting cumulative hazard function $H(x)$ against failure times on a hazard paper or a simple log-log paper.

$$H(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}$$

The cumulative hazard function is given below:

$$H(x) = \int h(x) = \left(\frac{x}{\eta}\right)^{\beta} \qquad (3)$$

We can transform (3) by taking the logarithm as follows

$$\ln H(x) = \beta\{\ln x - \ln \eta\}$$

$$\ln x = \frac{1}{\beta}\ln H(x) + \ln \eta \qquad (4)$$

From (4), We can then plot $\ln H(x)$ versus $\ln x$ using the following procedure:

1. Rank the failure times in ascending order.
2. For each failure, calculate $\Delta H_i = \frac{1}{(n+1)-1}$
3. For each failure, calculate H= $\Delta H_1 + \Delta H_2 + \cdots + \Delta H_i$
4. Plot $\ln H$ vs. $\ln x$.
5. Fit a straight line.

Upon completing the plotting, the estimated parameters will be as follows:

$$\beta = \left(\frac{x}{y}\right) = \frac{1}{slope}$$

At H=1, $\eta = x$

Analytical Methods

Due to the high probability of error in using graphical methods, we prefer to use the analytical methods. This is motivated by the availability of high-speed computer. In the following, we discuss some of the analytical methods use in estimating Weibull parameters.

Maximum Likelihood Estimator (MLE)

The method of maximum likelihood (Harter and Moore (1965a), Harter and Moore (1965b), and Cohen (1965) is a commonly used procedure because it has very desirable properties. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from a probability density function $f_x(x;\theta)$

where θ is an unknown parameter. The likelihood function of this random sample is the joint density of the $n$random variables and is a function of the unknown parameter. Thus

$$L = \prod_{i=1}^{n} f_{x_i}(x,\theta) \tag{5}$$

Is the likelihood function. The maximum likelihood estimator (MLE) of , say $\hat{\theta}$, is the value of θ that maximizes L or, equivalently, the logarithm of L. Often, but not always, the MLE of $\theta$is a solution of

$$\frac{d \log L}{d\theta}=0$$

where solutions that are not functions of the sample values $x_1, x_2, \ldots, x_n$are not admissible, nor are solutions which are not in the parameter space. Now, we are going to apply to MLE to estimate the Weibull parameters, namely the shapeand the scale parameters. Consider the Weibull pdf given in (1), then likelihood function will be

$$(L(x_1,x_2,\ldots\ldots,x_n;\beta,\eta) = \prod_{i=1}^{n}\left(\frac{\beta}{\eta}\right)\left(\frac{x_i}{\eta}\right)^{\beta-1} e^{-\left(\frac{x_i}{\eta}\right)^{\beta}} \tag{6}$$

On taking the logarithms of (6), differentiating with respect to β and η in turn and equating to zero, we obtain the estimating equations.

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln x_i - \frac{1}{\eta}\sum_{i=1}^{n} x_i^{\beta} \ln x_i = 0 \tag{7}$$

$$\frac{\partial \ln L}{\partial \eta} = -\frac{n}{\eta} + \frac{1}{\eta^2}\sum_{i=1}^{n} x^{\beta}{}_i = 0 \tag{8}$$

On eliminating η between these two equations and simplifying, we have

$$\frac{\sum_{i=1}^{n} x_i^{\beta} \ln x_i}{\sum_{i=1}^{n} x_i^{\beta}} - \frac{1}{\beta} - \frac{1}{n}\sum_{i=1}^{n} \ln x_i = 0 \tag{9}$$

Which may be solved to get the estimate of $\hat{\mu}_k = \beta$. This can be accomplished by the use of standard iterative procedures (i.e., Newton-Raphson method). Once β is determined, η can be estimated using equation (8) as

$$\eta = \frac{\sum_{i=1}^{n} x_i^{\beta}}{n} \tag{10}$$

Method of moments (MOM)

The method of moments is another technique commonly used in the field of parameter estimation. If the number $x_1, x_2, \ldots, x_n$ represent a set of data, then an unbiased estimator for the k^{th} origin moment is

$$\hat{m}_k = \frac{1}{n}\sum_{i=1}^{n} x_i^{k} \tag{11}$$

Where; $\hat{m}_k$ stands for the estimate of m_k. In Weibull distribution, the k^{th} moment readily follows from (1) as

$$\mu_k = \left(\frac{1}{\eta^{\beta}}\right)^{\frac{k}{\beta}} \Gamma\left(1 + \frac{k}{\beta}\right) \tag{12}$$

Where Γ signifies the gamma function

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx, (s > 0)$$

Then from (12), we can find the first second moments as follows

$$m_1 == \hat{\mu}_k = \left(\frac{1}{\eta^{\beta}}\right)^{\frac{k}{\beta}} \Gamma\left(1 + \frac{1}{\beta}\right) \tag{13}$$

$$m_2 = \mu_k^2 + \hat{\sigma}_k^2 = \left(\frac{1}{\eta}\right)^{\frac{2}{\beta}} \{\Gamma\left(1 + \frac{2}{\beta}\right) - [\Gamma\left(1 + \frac{1}{\beta}\right)]^2\} \tag{14}$$

When we divide m_2 by the square of m_1 , we get an expression which is a function of β only

$$\frac{\hat{\sigma}_k^2}{\mu_k^2} = \frac{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)}{\Gamma^2\left(1+\frac{1}{\beta}\right)} \tag{15}$$

On taking the square root of (15), we have the coefficient of variation

$$\text{cv} = \frac{\sqrt{\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma^2\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)} \tag{16}$$

Now, we can form a table for various cv by using (16) for different β values. In order to estimate β and η , we need to calculate the coefficient of variation $(cv)_d$ of the data on hand. Having done this, compare $(cv)_d$ with cv using the table. The corresponding β is the estimated one ($\hat{\beta}$). The scale parameter (η) can then be estimated using the following

$$\hat{\eta} = \{\bar{x}/\Gamma[(1/\hat{\beta}) + 1]\}^{\hat{\beta}}$$

Where $\bar{x}$ is the mean of the data.

Least square method (LSM)

The third estimation technique we shall discuss is known as the Least Squares Method. It is so commonly applied in engineering and Mathematics problems that is often not thought of as an estimation problem. We assume that a linear relation between the two variables (see section 1). For the estimation of Weibull parameters, we use the method of least square and we apply it to the results of section 1. Recall that

$$\ln\ln\left[\frac{1}{1-F(x)}\right] = \beta\ln x - \beta\ln\eta \qquad (17)$$

Equation (17) is a linear equation. Now, we can write

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\ln\{\ln\left[\frac{1}{\left(1-\frac{i}{n+1}\right)}\right]\} \qquad (18)$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n}\ln x_i \qquad (19)$$

$$\hat{\beta} = \frac{[\ n.\sum_{i=1}^{n}(\ln .x_i).\left(\ln\left\{\ln\left[\frac{1}{\left(1-\frac{i}{n+1}\right)}\right]\right\}\right)\}-\left\{\sum_{i=1}^{n}\ln\left(\ln\left[\frac{1}{\left(1-\frac{i}{n+1}\right)}\right]\right).\sum_{i=1}^{n}\ln x_i\right\}}{\left[n.\sum_{i=1}^{n}9\ln x_i\right)^2-\left\{\sum_{i=1}^{n}(\ln x_i\ \right\}^2} \qquad (20)$$

$$\hat{\eta} = e^{(\bar{y}-\frac{\bar{x}}{\hat{\beta}})} \qquad (21)$$

From equations (18)-(21), we can calculate the estimate of β and η.

Example:

Age-adjusted prevalence of type1 diabetes, 2 diabetes, and IFG is shown in Figure 2 according to the selected age groups. The prevalence of type 1 diabetes, type 2 diabetes, and IFG increased with age reaching its peak in the age group more than 65 years at 4.0% for type 1 diabetes, 40.6% for type 2 diabetes, and 29.5% - for IFG. For children aged 0-6 years, the prevalence of type 1 diabetes was found to be 0.4%, whereas that of IFG was found to be 2.8%; however, no cases of type 2 diabetes were discovered in this age group. In the age group from 7-18 years, type 1 diabetes remained at 0.4%, whereas type 2 diabetes at 5.2% and IFG at 6.4% which was almost similar to the age group from 19 - 24 years at 0.5%, 5.5%, and 6.7% for type 1 diabetes, type 2 age diabetes, and IFG, respectively. Abnormal glucose metabolism increased in the age group from 25-45 years, where the prevalence of type 1 diabetes reached 0.8%, type 2 diabetes 11.7%, and IFG 9.5%. This prevalence continued to increase reaching 2.5% for type 1 diabetes, 34.5% for type 2 diabetes, and 13.9% for IFG in the age group from 46-65 years.

Type 1 diabetes	0	0.0292	0.0292	0.0576	0.0851	0.1118	0.1357
Type 1 diabetes	0	0.0482	0.1647	0.1647	0.6812	1.1276	1.2452
Impaired fasting glucose (IFG)	0	0.1407	0.3051	0.3051	0.4548	0.5279	1.0803

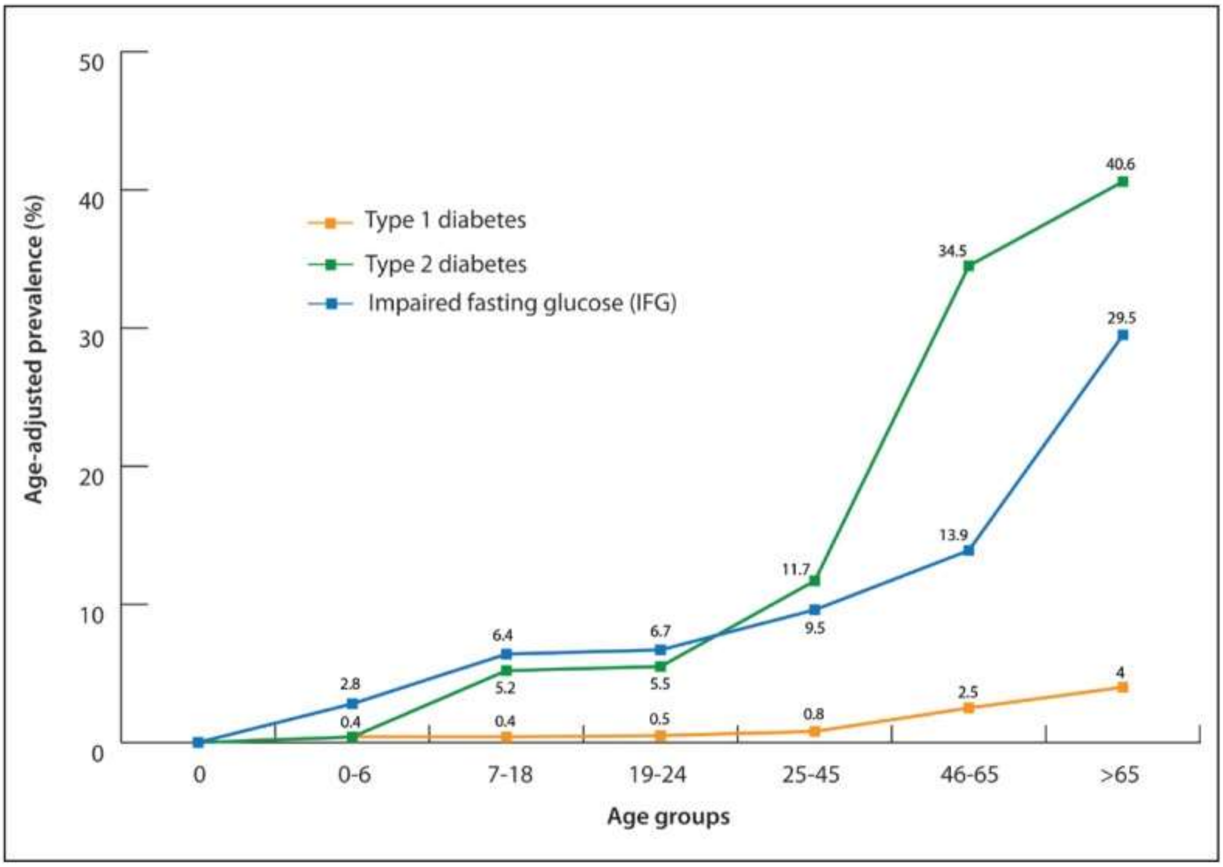

Figure: 1

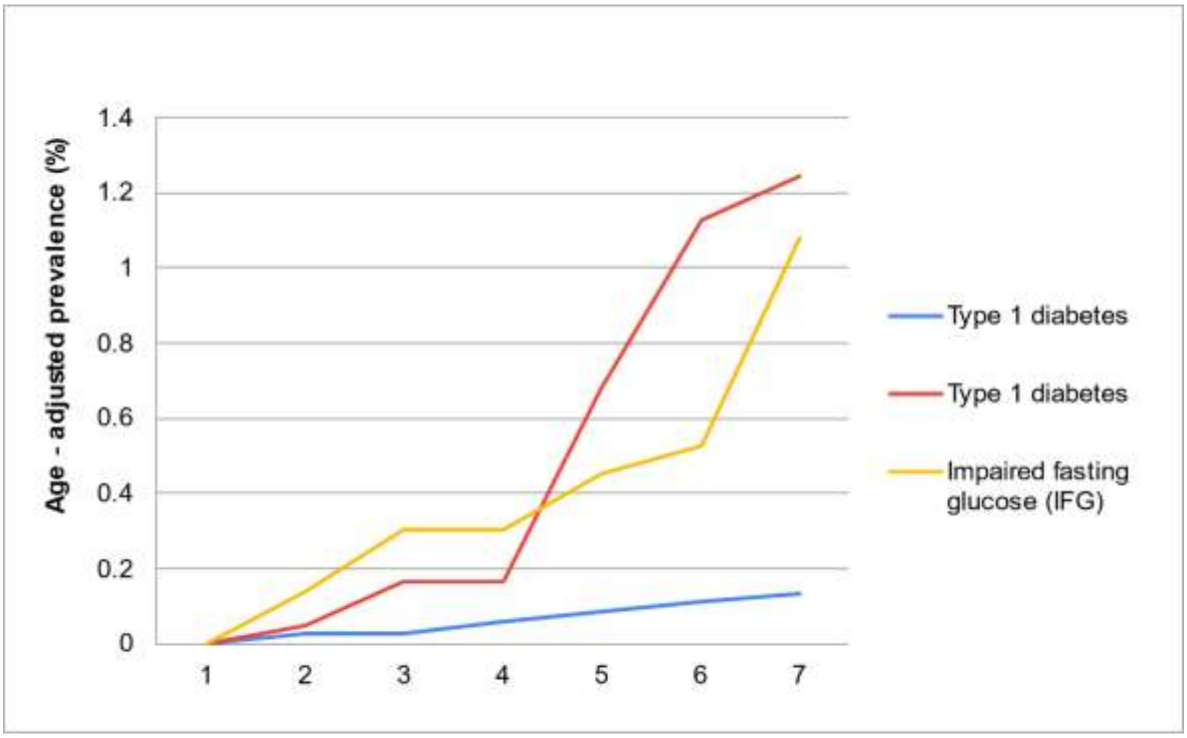

Figure: 2

Conclusion:

The Present study investigated the abnormal glucose metabolism has reached an epidemic state in Saudi Arabia. Where one-third of the population is affected and half of diabetic cases were unaware of their disease. We analysed using Weibull probability methods. Finally from fig(2), We conclude that the results coincides with the Mathematical and Medical report.

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